Random Access and Source-Channel Coding Error Exponents for Multiple Access Channels

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Abstract

New universal coding/decoding scheme for Random Access with collision detection is given in case of two senders. The result is used to give an achievable source-channel coding error exponent for Multiple-Access channels in case of independent sources.

1 Introduction

This paper addresses a version of the random access model of Luo, Epremides [6] and Wang, Luo [8], which is similar to the model studied for one-way channels by Csiszár [2]. In the terminology of this paper, in [2] the performance of a codebook library consisting of several constant composition codebooks with pre-determined rates has been analyzed. That result shows that it is possible to achieve universally the same error exponent for each codebook as the random-coding error exponent of this codebook alone. This theorem is used in [2] to give an achievable error exponent for joint source-channel coding (JSCC).

This paper generalized the mentioned results of [2] to (discrete memoryless) multiple-access channels (MACs). A two-senders random-access model is introduced, in which the senders have codebook libraries with constant composition codebooks for multiple rate choices. The error exponent of Liu and Hughes [5] for an individual codebook pair is shown to be universally achievable for each codebook pair in the codebook libraries, supplemented with collision detection in the sense of [6, 8]. In particular, a positive answer is given to the question in [6] whether the results there are still valid if the receiver does not know the channel. Moreover, an achievable JSCC error exponent for transmitting independent sources over a MAC is given, better than that achievable by separate source and channel coding.

Nazari, Anastasopulos, and Pradhan in [7] derive achievable error exponents for MAC's using α-decoding rule introduced for one-way channels in [3] by Csiszár and Körner. In the present paper a particular α-decoder is used. However, as the proofs follow closely [7], it can be seen that other choices could also be appropriate depending on actual assumptions on the analyzed models.

Note that, an other multiterminal generalization of the JSCC result in [2] appears in Zhong, Alajaji, Campbell [9]. We also mention that, this paper as [2], also has connections with the topic of unequal protection of messages, see for example Borade, Nakiboglu, Zheng [1].

2 Preliminaries, notation

Denote the set $\{1, 2, \ldots, M\}$ by $[M]$. The notation follows [4] and [7] whenever possible, for example, the following notations are used: $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, $\mathcal{P}(\mathcal{X}|\mathcal{U})$, $\mathcal{P}^n(\mathcal{X})$, T_P^n, $T^n_{P_{X|U}}(\mathbf{u})$. Let $\mathcal{P}^n(\mathcal{X}|P_U)$ be the collection of all conditional distribution $V_{X|U}$ for wich there exists an $\mathbf{x} \in T^n_{V_{X|U}}(\mathbf{u})$ for some $\mathbf{u} \in T^n_{P_U}$.

Denote $\mathrm{H}_V(X, Y)$, $\mathrm{H}_V(X, Y, U)$, $\mathrm{I}_V(X \wedge Y)$ etc. the entropy and mutual information when the random variables X, Y, U have joint distribution V_{XY}, V_{XYU} etc. Denote $\mathrm{I}(\mathbf{x} \wedge \mathbf{y})$, $\mathrm{H}(\mathbf{x}, \mathbf{y})$ etc. the information quantities $\mathrm{I}_V(X \wedge Y)$, $\mathrm{H}_V(X, Y)$ etc. with V_{XY} equal to the type of $(\mathbf{x}, \mathbf{y})$. If V_{XYZU} is a multivariate distribution on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{U}$ then V_{XYU}, V_{XU}, V_{YU} etc. denote the marginal distributions respectively. Moreover, we define the following multi-information (see [4] exercise 3.9. and [5], but different from Yeung's notation):

$$\begin{aligned}\mathrm{I}(X_1 \wedge X_2 \wedge \cdots \wedge X_N|Y) \triangleq\ & \mathrm{H}(X_1|Y) + \mathrm{H}(X_2|Y) + \ldots \\ & + \mathrm{H}(X_N|Y) - \mathrm{H}(X_1, X_2, \ldots, X_N|Y) \end{aligned} \tag{1}$$

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Given a MAC $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, the pentagon

$$\left\{ \begin{array}{l} (R_1, R_2) : 0 \le R_1 \le \mathrm{I}(X \wedge Z|Y, U), \\ 0 \le R_2 \le \mathrm{I}(Y \wedge Z|X, U), R_1 + R_2 \le \mathrm{I}(X, Y \wedge Z|U) \end{array} \right\},$$

where U, X, Y, Z have joint distribution equal to $P_U P_{X|U} P_{Y|U} W$, is denoted by $C[W, P_U, P_{X|U}, P_{Y|U}]$. The union of these pentagons, i.e., the capacity region of the MAC W, is denoted by $C(W)$.

3 Random Access with collision detection

In this model two transmitters try to communicate over a MAC W with one common receiver. The channel W is unknown to the senders and may also be unknown to the receiver (but see Remark 2). Both senders have multiple codebooks with block length n. We assume that the codewords' conditional tpye on an auxiliary sequence $\mathbf{u}$ is fixed in all codebooks, however this common type can vary from codebook to codebook.

Definition 1. *Let a finite set $\mathcal{U}$, a sequence $\mathbf{u} \in \mathcal{U}^n$ of type $P_U \in \mathcal{P}^n(\mathcal{U})$, positive integers M_1 and M_2, conditional distributions $\{P^i_{X|U} \in \mathcal{P}^n(\mathcal{X}|P_U), i \in [M_1]\}$, $\{P^j_{Y|U} \in \mathcal{P}^n(\mathcal{Y}|P_U), j \in [M_2]\}$, rates $\{R^i_1, i \in [M_1]\}$ and $\{R^j_2, j \in [M_2]\}$ be given parameters. A constant composition codebook library pair of length n with the above parameters is a pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ consist of constant composition codebooks $(A^1, \dots, A^{M_1})$ resp. $(B^1, \dots, B^{M_2})$ such that $A^i = \{\mathbf{x}^i_1, \mathbf{x}^i_2, \dots \mathbf{x}^i_{N^i_1}\}$ and $B^j = \{\mathbf{y}^j_1, \mathbf{y}^j_2, \dots \mathbf{y}^j_{N^j_2}\}$ with $\mathbf{x}^i_a \in \mathcal{T}^n_{P^i_{X|U}}(\mathbf{u})$ and $\mathbf{y}^j_b \in \mathcal{T}^n_{P^j_{Y|U}}(\mathbf{u})$, $i \in [M_1]$, $j \in [M_2]$, $N^i_1 = \left\lfloor e^{nR^i_1} \right\rfloor$, $N^j_2 = \left\lfloor e^{nR^j_2} \right\rfloor$, $a \in [N^i_1]$, $b \in [N^j_2]$.*

Before sending messages, each transmitter choses one of its codebooks independently from the other sender. Denote this selection by $(i, j) \in [M_1] \times [M_2]$. The transmitters do not share the result of their selections with each other, neither with the receiver. The senders send codewords $\mathbf{x}^i_a$, $\mathbf{x}^j_b$. The decoder output $\hat{\mathbf{m}}$ is either a quadruple (i, a, j, b) or "collision". The receiver is required to decode quadruple (i, a, j, b) if the rate pair (R^i_1, R^j_2) of the choosen codebooks is in the interior of $C[W, P_U, P_{X^i|U}, P_{Y^j|U}]$ and to declare "collision" otherwise. Hence, two types of error are defined.

Definition 2. *For the codebooks (A^i, B^j), the average decoding error probability is*

$$Err_d(i, j) = \frac{1}{N^i_1 N^j_2} \sum_{\mathbf{m} \in A^i \times B^j} Pr\{\hat{\mathbf{m}} \neq \mathbf{m} | \mathbf{m} \text{ is sent}\},$$

and the average collision declaration error probability is

$$Err_c(i, j) = \frac{\sum_{\mathbf{m} \in A^i \times B^j} Pr\{\hat{\mathbf{m}} \neq \text{"collision"} | \mathbf{m} \text{ is sent}\}}{N^i_1 N^j_2}.$$

To state our basic theorem we need the following notions from [5]; the index HL refers to the authors of [5].

$$\begin{aligned} &\mathcal{V}_{HL} = \mathcal{V}_{HL}(W, P_U, P_{X|U}, P_{Y|U}) \\ &\qquad \triangleq \left\{ V_{UXYZ} : V_{UX} = P_U P_{X|U}, V_{UY} = P_U P_{Y|U} \right\} && (2) \\ &EX_{HL}(R_1, R_2, W, P_U, P_{X|U}, P_{Y|U}) \\ &\qquad \triangleq \min_{V_{UXYZ} \in \mathcal{V}_{HL}} [D(V_{Z|XYU} || W | V_{XYU}) \\ &\qquad\qquad + I_V(X \wedge Y|U) + |I_V(X \wedge YZ|U) - R_1|^+] && (3) \\ &EY_{HL}(R_1, R_2, W, P_U, P_{X|U}, P_{Y|U}) \\ &\qquad \triangleq \min_{V_{UXYZ} \in \mathcal{V}_{HL}} [D(V_{Z|XYU} || W | V_{XYU}) \\ &\qquad\qquad + I_V(X \wedge Y|U) + |I_V(Y \wedge XZ|U) - R_2|^+] && (4) \\ &EXY_{HL}(R_1, R_2, W, P_U, P_{X|U}, P_{Y|U}) \\ &\qquad \triangleq \min_{V_{UXYZ} \in \mathcal{V}_{HL}} [D(V_{Z|XYU} || W | V_{XYU}) && (5) \\ &\qquad\qquad + I_V(X \wedge Y|U) + |I_V(X \wedge Y \wedge Z|U) - R_1 - R_2|^+] \\ &E_{HL}(R_1, R_2, W, P_U, P_{X|U}, P_{Y|U}) \\ &\qquad \triangleq \min\{EX_{HL}, EY_{HL}, EXY_{HL}\} && (6) \end{aligned}$$

Theorem 1 shows that the error exponent of [5] for an individual codebook pair is achievable for this general setting.

Theorem 1. *For each n let constant composition random access codebook library parameters as in Definition 1 be given with a common set $\mathcal{U}$ and with $\frac{1}{n}logM_1 \to 0$, $\frac{1}{n}logM_2 \to 0$ as $n \to \infty$. Then there exist a sequence $\delta_n(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|, \{M_1\}_{n=1}^{\infty}, \{M_2\}_{n=1}^{\infty}) \to 0$ and for each n a constant composition codebook-library pair $(\mathcal{A}, \mathcal{B})$ with the given parameters, and decoder mappings with the following properties:*

(i) For all $(i,j) \in [M_1] \times [M_2]$

$$Err_d(i,j) \leq e^{-n(E_{HL}(R_1^i, R_2^j, W, P_U, P^i_{X|U}, P^j_{Y|U}) - \delta_n)}. \tag{7}$$

(ii) If (R_1^i, R_2^j) is not in the interior of $C[W, P_U, P^i_{X|U}, P^j_{Y|U}]$ then

$$Err_c(i,j) < \delta_n. \tag{8}$$

Remark 1. Note that in part (i) of Theorem 1 $E_{HL}(R_1^i, R_2^j, W, P_U, P^i_{X|U}, P^j_{Y|U}) > 0$ iff (R_1^i, R_2^j) is in the interior of $C[W, P_U, P^i_{X|U}, P^j_{Y|U}]$.

The next packing lemma is an extension of Lemma 4 in [7] for this multiple codebooks setting, it provides the appropriate codebook library pair for Theorem 1. Note that multi information is used instead of the original notation of [7].

Lemma 1. *Let a sequence of constant composition random access codebook library parameters be given as in Theorem 1. Then there exist a sequence $\delta_n^{'}(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, \{M_1\}_{n=1}^{\infty}, \{M_2\}_{n=1}^{\infty}) \to 0$ and for each n a constant composition codebook-library pair $(\mathcal{A}, \mathcal{B})$ with the given parameters such that for any $(i,k) \in [M_1]^2$ and $(j,l) \in [M_2]^2$ and for all $V_{UX\hat{X}Y\hat{Y}U} \in \mathcal{P}^n(\mathcal{U} \times \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Y})$:*

$$\begin{aligned}
K^{i,j}_{k,l}[V_{UXY}] &\leq 2^{-n(\mathrm{I}_V(X \wedge Y|U) - R_1^i - R_2^j - \delta_n^{'})} \\
K^{i,j}_{k,l}[V_{UX\hat{X}Y}] &\leq 2^{-n(\mathrm{I}_V(X \wedge \hat{X} \wedge Y|U) - R_1^i - R_2^j - R_1^k - \delta_n^{'}} \\
K^{i,j}_{k,l}[V_{UXY\hat{Y}}] &\leq 2^{-n(\mathrm{I}_V(X \wedge Y \wedge \hat{Y}|U) - R_1^i - R_2^j - R_2^l - \delta_n^{'})} \\
K^{i,j}_{k,l}[V_{UX\hat{X}Y\hat{Y}}] &\leq 2^{-n(\mathrm{I}_V(X \wedge \hat{X} \wedge Y \wedge \hat{Y}|U) - R_1^i - R_2^j - R_1^k - R_2^l - \delta_n^{'})},
\end{aligned}$$

where

$$K^{i,j}_{k,l}[V_{UXY}] \triangleq \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} \mathbb{1}_{\mathcal{T}_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_a^i, \mathbf{y}_b^j) \tag{9}$$

$$K^{i,j}_{k,l}[V_{UX\hat{X}Y}] \triangleq \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} \sum_{\substack{c=1 \\ c \neq a \text{ if } i=k}}^{N_1^k} \mathbb{1}_{\mathcal{T}_{V_{UX\hat{X}Y}}}(\mathbf{u}, \mathbf{x}_a^i, \mathbf{x}_c^k, \mathbf{y}_b^j) \tag{10}$$

$$K^{i,j}_{k,l}[V_{UXY\hat{Y}}] \triangleq \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} \sum_{\substack{d=1 \\ d \neq b \text{ if } j=l}}^{N_2^l} \mathbb{1}_{\mathcal{T}_{V_{UXY\hat{Y}}}}(\mathbf{u}, \mathbf{x}_a^i, \mathbf{y}_b^j, \mathbf{y}_d^l) \tag{11}$$

$$\begin{aligned}
&K^{i,j}_{k,l}[V_{UX\hat{X}Y\hat{Y}}] \triangleq \\
&\quad \triangleq \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} \sum_{\substack{c=1 \\ c \neq a \text{ if } i=k}}^{N_1^k} \sum_{\substack{d=1 \\ d \neq b \text{ if } j=l}}^{N_2^l} \mathbb{1}_{\mathcal{T}_{V_{UXY\hat{Y}}}}(\mathbf{u}, \mathbf{x}_a^i, \mathbf{x}_c^k, \mathbf{y}_b^j, \mathbf{y}_d^l).
\end{aligned} \tag{12}$$

Sketch of proof: Let $(\mathcal{A}, \mathcal{B})$ be a random constant composition codebook library pair, i. e for all $i \in [M_1], j \in [M_2]$ the codewords of A^i, B^j are chosen independently and uniformly from $\mathcal{T}^n_{P^i_{X|U}}(\mathbf{u})$ and $\mathcal{T}^n_{P^j_{Y|U}}(\mathbf{u})$ respectively. Let T be the following random variable:

$$\begin{aligned}
T = &\sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \sum_{k=1}^{M_1} \sum_{l=1}^{M_2} \sum_{\substack{V_{UX\hat{X}Y\hat{Y}} \in \\ \mathcal{P}^n(\mathcal{U} \times \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Y})}} \Big[K^{i,j}_{k,l}[V_{UXY}] \\
&\cdot 2^{n(\mathrm{I}_V(X \wedge Y|U) - R_1^i - R_2^j - \delta_n^{'})} \\
&+ K^{i,j}_{k,l}[V_{UX\hat{X}Y}] 2^{n(\mathrm{I}_V(X \wedge \hat{X} \wedge Y|U) - R_1^i - R_2^j - R_1^k - \delta_n^{'})} \\
&+ K^{i,j}_{k,l}[V_{UXY\hat{Y}}] 2^{n(\mathrm{I}_V(X \wedge Y \wedge \hat{Y}|U) - R_1^i - R_2^j - R_2^l - \delta_n^{'})} \\
&+ K^{i,j}_{k,l}[V_{UX\hat{X}Y\hat{Y}}] 2^{n(\mathrm{I}_V(X \wedge \hat{X} \wedge Y \wedge \hat{Y}|U) - R_1^i - R_2^j - R_1^k)} \\
&\cdot 2^{n(-R_2^l - \delta_n^{'})} \Big].
\end{aligned} \tag{13}$$

Using basic properties of types (similarly as in [2, 7]) it can be seen that we can choose $\delta_n^{'} \to 0$ such that $\mathbb{E}[T] < 1$. Hence there exists a realization of the codebook library pair with $T < 1$. Taking into account the positivity of the terms of T, the lemma is proved. ■

Sketch of proof of Theorem 1: Lemma 1 provides the appropriate constant composition codebook-library pair $(\mathcal{A}, \mathcal{B})$. To construct the decoder, define $\alpha : \mathcal{P}(\mathcal{U} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}) \to R$ by $\alpha(V_{UXYZ}) = \mathrm{I}_V(X \wedge Y \wedge Z|U)$. In the first stage of decoding, the receiver determine the indices $\hat{k} \in [M_1]$, $\hat{l} \in [M_2]$, $\hat{a} \in [N_1^{\hat{k}}]$, $\hat{b} \in [N_2^{\hat{l}}]$ which maximize $\alpha(\mathbf{u}, \mathbf{x}_a^k, \mathbf{y}_b^l, \mathbf{z}) - R_1^k - R_2^l$, where $\mathbf{z}$ denotes the output sequence and α is evaluated on the joint type of $(\mathbf{u}, \mathbf{x}_a^k, \mathbf{y}_b^l, \mathbf{z})$. In the second stage,to deal with collisions, the decoder checks the following three inequalities:

$$I(\mathbf{x}_{\hat{a}}^{\hat{k}} \wedge \mathbf{y}_{\hat{b}}^{\hat{l}} \wedge \mathbf{z}|\mathbf{u}) - R_1^{\hat{k}} - R_2^{\hat{l}} > \eta_n \tag{14}$$

$$I(\mathbf{x}_{\hat{a}}^{\hat{k}} \wedge \mathbf{y}_{\hat{b}}^{\hat{l}}, \mathbf{z}|\mathbf{u}) - R_1^{\hat{k}} > \eta_n \tag{15}$$

$$I(\mathbf{y}_{\hat{a}}^{\hat{k}} \wedge \mathbf{x}_{\hat{b}}^{\hat{l}}, \mathbf{z}|\mathbf{u}) - R_2^{\hat{l}} > \eta_n, \tag{16}$$

where $\eta_n(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|, \{M_1\}_{n=1}^{\infty}, \{M_2\}_{n=1}^{\infty}) \to 0$ is an appropriately chosen positive sequence. If the above three inequalities are fullfilled then the decoder decodes $\mathbf{x}_{\hat{a}}^{\hat{k}}$, $\mathbf{y}_{\hat{b}}^{\hat{l}}$ as the codewords sent, if at least one of them is not fullfilled, then the decoder reports "collision".

Some details about the necessary calculations can be found in the Appendix. ■

Remark 2. Other α-decoders can be also used; if the receiver knows the channel W, the α function can depend on W. Note also that for the sake of brevity the expurgation method for multiple-access channel in [7] is not used in this paper. However, it is possible to prove an expurgated version of Lemma 1 which leads larger achievable error exponent for small rates.

4 Source-channel coding

Let two independent discrete memoryless sources (DMS) Q_1,Q_2 with alphabets $\mathcal{S}_1, \mathcal{S}_2$ be given. We want to transmit these sources over MAC W. It is assumed that the sources Q_1 and Q_2 and the channel W are known by the encoders, however, not known by the decoder.

Definition 3. *A source-channel code of length n is a mapping triple (f_1, f_2, φ) with encoders $f_1 : \mathcal{S}_1^n \to \mathcal{X}^n$, $f_2 : \mathcal{S}_2^n \to \mathcal{Y}^n$ and decoder $\varphi : \mathcal{Z}^n \to \mathcal{S}_1^n \times \mathcal{S}_2^n$.*

Definition 4. *The error of a source-channel code (f_1, f_2, φ) of length n is defined by*

$$Err(f_1, f_2, \varphi) = \sum_{\substack{(\mathbf{s}_1, \mathbf{s}_2) \in \\ \mathcal{S}_1^n \times \mathcal{S}_2^n}} Q_1^n(\mathbf{s}_1) Q_2^n(\mathbf{s}_2) p_e(\mathbf{s}_1, \mathbf{s}_2), \tag{17}$$

where

$$p_e(\mathbf{s}_1, \mathbf{s}_2) = W^n(\{\mathbf{z} \in \mathcal{Z}^n : \varphi(\mathbf{z}) \neq (\mathbf{s}_1, \mathbf{s}_2)\}|f_1(\mathbf{s}_1), f_2(\mathbf{s}_2)).$$

We assume that $(H(Q_1), H(Q_2))$ is in the interior of $C(W)$. In this case Q_1,Q_2 can be reliable transmitted over channel W, for example, by separate source and channel coding. Regarding error exponents, by separate coding the exponent $Es_{HL}(Q_1, Q_2, W)$ is achievable, where

$$Es_{HL}(Q_1, Q_2, W) \triangleq \max_{\substack{0 \le R_1 \le log|S_1| \\ 0 \le R_2 \le log|S_2|}} \min \Big[e_1(R_1, Q_1), \\ e_2(R_2, Q_2), E_{HL}(R_1, R_2, W) \Big], \tag{18}$$

$e_1(R_1, Q_1)$,$e_2(R_2, Q_2)$ are the source reliability functions

$$e_i(R_i, Q_i) \triangleq \min_{P : \mathrm{H}(P) \ge R_i} D(P \| Q_i), \quad i \in \{1, 2\}, \tag{19}$$

and

$$E_{HL}(R_1, R_2, W) \triangleq \\ \triangleq \sup_{\substack{\mathcal{U} \\ P_U \in \mathcal{P}(\mathcal{U})}} \sup_{\substack{P_{X|U} \in \mathcal{P}(\mathcal{X}|\mathcal{U}) \\ P_{Y|U} \in \mathcal{P}(\mathcal{Y}|\mathcal{U})}} E_{HL}[R_1, R_2, W, P_U, P_{X|U}, P_{Y|U}]. \tag{20}$$

Note that $Es_{HL}(Q_1, Q_2, W) > 0$. Of course, (18) could be improved replacing $E_{HL}(R_1, R_2, W)$ by the reliability function of channel W which, however, is not known in general.

In this section a larger exponent then Es_{HL} is given using JSCC.

For arbitrary $\mathcal{U}$ and $P_U \in \mathcal{P}(\mathcal{U})$ let $G_1(\mathcal{U})$ and $G_2(\mathcal{U})$ be the set of all continous mappings $[0, log|S_1|] \to \mathcal{P}(\mathcal{X}|\mathcal{U})$ and $[0, log|S_2|] \to \mathcal{P}(\mathcal{Y}|\mathcal{U})$ respectively, and define

$$Ej(Q_1, Q_2, W) \triangleq \sup_{\substack{\mathcal{U} \\ P_U \in \mathcal{P}(\mathcal{U})}} \sup_{\substack{g_1 \in G_1(\mathcal{U}) \\ g_2 \in G_2(\mathcal{U})}} Ej(Q_1, Q_2, W, P_U, g_1, g_2)$$

where

$$Ej(Q_1, Q_2, W, P_U, g_1, g_2) \triangleq \min_{\substack{0 \le R_1 \le log|S_1| \\ 0 \le R_2 \le log|S_2|}} \Big[e_1(R_1) + e_2(R_2) + E_{HL}(R_1, R_2, W, P_U, g_1(R_1), g_2(R_2)) \Big]. \quad (21)$$

Proposition 2. $Es_{HL}(Q_1, Q_2, W) \le Ej(Q_1, Q_2, W)$.

Proof: Restricting the suprerum to constant functions g_1, g_2 in (21), we see that:

$$Ej(Q_1, Q_2, W) \ge \sup_{\substack{\mathcal{U} \\ P_U \in \mathcal{P}(\mathcal{U})}} \sup_{\substack{P_{X|U} \in \mathcal{P}(\mathcal{X}|\mathcal{U}) \\ P_{Y|U} \in \mathcal{P}(\mathcal{Y}|\mathcal{U})}} \min_{\substack{0 \le R_1 \le log|S_1| \\ 0 \le R_2 \le log|S_2|}} \Big[e_1(R_1) + e_2(R_2) + E_{HL}(R_1, R_2, W, P_U, P_{X|U}, P_{Y|U})]. \quad (22)$$

Using the definition of $e_1(R_1)$, $e_2(R_2)$ and $E_{HL}(R_1, R_2, W, P_U, P_{X|U}, P_{Y|U})]$ it can be easily seen that the right-hand side of (22) is greater than or equal to $Es_{HL}(Q_1, Q_2, W)$. ■

Remark 3. Actually, the strict inequality holds, except in very special cases.

The following theorem shows that $Ej(Q_1, Q_2, W)$ is an achievable error exponent for this source-channel coding scenario, hence JSCC leads to larger exponent than separate source and channel coding. More exactly, we show that for any choice of P_U, g_1, g_2, the exponent $Ej(Q_1, Q_2, W, P_U, g_1, g_2)$ is achievable even if the senders do not know the sources and the channel; if they do know them, they can optimize in P_U, g_1, g_2, to achieve $Ej(Q_1, Q_2, W)$.

Theorem 3. *Let $\mathcal{U}$, $P_U \in \mathcal{P}(\mathcal{U})$, $g_1 \in G_1(\mathcal{U})$ and $g_2 \in G_2(\mathcal{U})$ be given. There exist a source-channel code for each n and a sequence $\nu_n(|\mathcal{S}_1|, |\mathcal{S}_2|, |\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|) \to 0$ with*

$$Err(f_1, f_2, \varphi) \le 2^{-n(Ej(Q_1, Q_2, W, P_U, g_1, g_2) - \nu_n)}. \quad (23)$$

Sketch of proof: Approximate uniformly P_U, g_1, g_2 by sequences $P_U[n] \in \mathcal{P}^n(\mathcal{U})$, $g_1[n] : [0, log|S_1|] \to \mathcal{P}(\mathcal{X}|P_U(n))$, $g_2[n] : [0, log|S_2|] \to \mathcal{P}(\mathcal{Y}|P_U(n))$.

Let $\mathbf{u} \in \mathcal{T}^n_{P_U[n]}$ be arbitrary sequence. Choose $M_1 = |\mathcal{P}^n(S_1)|$ and $M_2 = |\mathcal{P}^n(S_2)|$. Let $P_1^1, P_1^2, \ldots, P_1^{M_1}$ and $P_2^1, P_2^2, \ldots, P_2^{M_2}$ denote all possible types from $\mathcal{P}^n(\mathcal{S}_1)$ and $\mathcal{P}^n(\mathcal{S}_2)$ respectively. For all $i \in [M_1]$, $j \in [M_2]$ let N_1^i and N_2^j be equal to $|\mathcal{T}^n_{P_1^i}|$ and $|\mathcal{T}^n_{P_2^j}|$ respectively, and let $P^i_{X|U}$ and $P^j_{Y|U}$ be equal to $g_1[n](N_1^i)$, $g_2[n](N_2^j)$ respectively. Applying Theorem 1 with these parameters consider the resulting codebook library pair $(\mathcal{A}, \mathcal{B})$ and the decoder mapping ϕ satisfying (7) for all $(i, j) \in [M_1] \times [M_2]$.

Let $f_1 : \mathcal{S}_1^n \to \mathcal{X}^n$ and $f_2 : \mathcal{S}_2^n \to \mathcal{Y}^n$ be the mappings which map each $\mathcal{T}^n_{P_1^i}$ and $\mathcal{T}^n_{P_2^j}$ to A^i and B^j respectively. Let $\varphi : \mathcal{Z}^n \to \mathcal{S}_1^n \times \mathcal{S}_2^n$ be the mapping which first determines a codeword pair from $(\mathcal{A}, \mathcal{B})$ using ϕ, then uses the inverse of f_1 and f_2 to determine the source sequences. The crucial step is the following equation

$$Err(f_1, f_2, \varphi) = \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} Q_1^n(\mathcal{T}^n_{P_1^i}) Q_2^n(\mathcal{T}^n_{P_2^j}) \cdot \frac{1}{|\mathcal{T}^n_{P_1^i}|} \frac{1}{|\mathcal{T}^n_{P_2^j}|} \sum_{\mathbf{s}_1 \in \mathcal{T}^n_{P_1^i}} \sum_{\mathbf{s}_2 \in \mathcal{T}^n_{P_2^j}} p_e(\mathbf{s}_1, \mathbf{s}_2) \quad (24)$$

Note that the second line of (24) is $Err_d(i, j)$ in the terminology of Theorem 1. Hence substituting (7) into (24) and using (19) and standard proprties of types, this theorem is proved. ■

Remark 4. Analogously to Lemma 2 of [2], it can be shown that the error exponent can not be greater than

$$\min_{\substack{0 \le R_1 \le log|S_1| \\ 0 \le R_2 \le log|S_2|}} [e_1(R_1) + e_2(R_2) + E(R_1, R_2, W)]$$

where $E(R_1, R_2, W)]$ is the reliability function of channel W.

[Sketch of proof of Theorem 1] Let us define the following sets for all $i \in [M_1]$, $j \in [M_2]$, $a \in [N_1^i]$, $b \in [N_2^j]$:

$$D_{a,b}^{i,j} \triangleq \left\{ \begin{array}{l} \mathbf{z} : \alpha(\mathbf{u}, \mathbf{x}_a^i, \mathbf{y}_b^j, \mathbf{z}) - R_1^i - R_2^j \\ \qquad \geq \alpha(\mathbf{u}, \mathbf{x}_c^k, \mathbf{y}_d^l, \mathbf{z}) - R_1^k - R_2^l, \text{for all} \\ \quad k \in [M_1], l \in [M_2], c \in [N_1^k], d \in [N_2^l] \end{array} \right\} \tag{25}$$

$$O_{a,b}^{i,j} \triangleq \left\{ \begin{array}{l} \mathbf{z} : \mathrm{I}(\mathbf{x}_a^i \wedge \mathbf{y}_b^j \wedge \mathbf{z}|\mathbf{u}) - R_1^i - R_2^j \leq \eta_n \text{ or} \\ \quad \mathrm{I}(\mathbf{x}_a^i \wedge \mathbf{y}_b^j, \mathbf{z}|\mathbf{u}) - R_1^i \leq \eta_n \text{ or} \\ \quad \mathrm{I}(\mathbf{y}_b^j \wedge \mathbf{x}_a^i, \mathbf{z}|\mathbf{u}) - R_2^j \leq \eta_n \end{array} \right\} \tag{26}$$

Then for all $(i, j) \in [M_1] \times [M_2]$:

$$\begin{aligned} Err_d(i,j) \leq{}& \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(O_{a,b}^{i,j} | \mathbf{x}_a^i, \mathbf{y}_b^j \right) \\ &+ \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(\bigcup_{\substack{c=1 \\ c \neq a}}^{N_1^i} D_{c,b}^{i,j} | \mathbf{x}_a^i, \mathbf{y}_b^j \right) \\ &+ \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(\bigcup_{\substack{d=1 \\ d \neq b}}^{N_2^j} D_{a,d}^{i,j} | \mathbf{x}_a^i, \mathbf{y}_b^j \right) \\ &+ \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(\bigcup_{\substack{c=1 \\ c \neq a}}^{N_1^i} \bigcup_{\substack{d=1 \\ d \neq b}}^{N_2^j} D_{c,d}^{i,j} | \mathbf{x}_a^i, \mathbf{y}_b^j \right) \\ &+ \sum_{\substack{k=1 \\ k \neq i}}^{M_1} \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(\bigcup_{c=1}^{N_1^k} D_{c,b}^{k,j} | \mathbf{x}_a^i, \mathbf{y}_b^j \right) \\ &+ \sum_{\substack{l=1 \\ l \neq j}}^{M_2} \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(\bigcup_{d=1}^{N_2^l} D_{a,d}^{i,l} | \mathbf{x}_a^i, \mathbf{y}_b^j \right) \\ &+ \sum_{\substack{k=1 \\ k \neq i}}^{M_1} \sum_{\substack{l=1 \\ l \neq j}}^{M_2} \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \left(\bigcup_{c=1}^{N_1^k} \bigcup_{d=1}^{N_2^l} D_{c,d}^{k,l} | \mathbf{x}_a^i, \mathbf{y}_b^j \right). \end{aligned} \tag{27}$$

For the sake of brevity, we introduce the following notations for the terms of the right-hand side of equation (27):

$$\begin{aligned} Err_d(i,j) \leq{}& th^{i,j} + errorX_{i,j}^{i,j} + errorY_{i,j}^{i,j} \\ &+ errorXY_{i,j}^{i,j} + \sum_{\substack{k=1 \\ k \neq i}}^{M_1} errorX_{k,j}^{i,j} \\ &+ \sum_{\substack{l=1 \\ l \neq j}}^{M_2} errorY_{i,l}^{i,j} + \sum_{\substack{k=1 \\ k \neq i}}^{M_1} \sum_{\substack{l=1 \\ l \neq j}}^{M_2} errorXY_{k,l}^{i,j}. \end{aligned} \tag{28}$$

Let us define the following expressions for all $i \in [M_1]$, $j \in [M_2]$, $k \in [M_1]$, $l \in [M_2]$:

$$\mathcal{VX}_{k,l}^{i,j} \triangleq \begin{Bmatrix} V_{UXY\tilde{X}Z} : \\ \alpha(V_{UXYZ}) - R_1^i \leq \alpha(V_{U\tilde{X}YZ}) - R_1^k, \\ V_{UX} = P_U P_{X|U}^i, \quad V_{U\tilde{X}} = P_U P_{X|U}^k, \\ V_{UY} = P_U P_{Y|U}^j \end{Bmatrix} \tag{29}$$

$$\mathcal{VY}_{k,l}^{i,j} \triangleq \begin{Bmatrix} V_{UXY\tilde{Y}Z} : \\ \alpha(V_{UXYZ}) - R_2^j \leq \alpha(V_{UX\tilde{Y}Z}) - R_2^l, \\ V_{UX} = P_U P_{X|U}^i, V_{UY} = P_U P_{Y|U}^j, \\ V_{U\tilde{Y}} = P_U P_{Y|U}^l. \end{Bmatrix} \tag{30}$$

$$\mathcal{VXY}_{k,l}^{i,j} \triangleq \left\{\begin{array}{l} V_{UXY\tilde{X}\tilde{Y}Z}: \\ \alpha(V_{UXYZ}) - R_1^i - R_2^j \\ \qquad \leq \alpha(V_{U\tilde{X}\tilde{Y}Z}) - R_1^k - R_2^l, \\ V_{UX} = P_U P_{X|U}^i, \quad V_{U\tilde{X}} = P_U P_{X|U}^k, \\ V_{UY} = P_U P_{Y|U}^j, \quad V_{U\tilde{Y}} = P_U P_{Y|U}^l. \end{array}\right\} \tag{31}$$

$$\begin{aligned} EX_{k,l}^{i,j} \triangleq & \min_{V_{UXY\tilde{X}Z} \in \mathcal{VX}_{k,l}^{i,j}} D(V_{Z|XYU}||W|V_{XYU}) \\ & + \mathrm{I}_V(X \wedge Y|U) + |\,\mathrm{I}_V(\tilde{X} \wedge XYZ|U) - R_1^i|^+ \qquad (32) \\ EY_{k,l}^{i,j} \triangleq & \min_{V_{UXY\tilde{Y}Z} \in \mathcal{VY}_{k,l}^{i,j}} D(V_{Z|XYU}||W|V_{XYU}) \\ & + \mathrm{I}_V(X \wedge Y|U) + |\,\mathrm{I}_V(\tilde{Y} \wedge XYZ|U) - R_2^j|^+ \qquad (33) \\ EXY_{k,l}^{i,j} \triangleq & \min_{V_{UXY\tilde{X}\tilde{Y}Z} \in \mathcal{VXY}_{k,l}^{i,j}} D(V_{Z|XYU}||W|V_{XYU}) \\ & + \mathrm{I}_V(X \wedge Y|U) + \Big|\,\mathrm{I}_V(\tilde{X}\tilde{Y} \wedge XYZ|U) \\ & + \mathrm{I}_V(\tilde{X} \wedge \tilde{Y}|U) - R_1^i - R_2^j\Big|^+ \qquad (34) \end{aligned}$$

Relating the error probailities to packing functions (9)-(12) as in [7] gives:

$$\begin{gathered} errorX_{k,l}^{i,j} \leq 2^{-n(EX_{k,l}^{i,j} - \delta_n^{''})}, \quad errorY_{k,l}^{i,j}, \leq 2^{-n(EY_{k,l}^{i,j} - \delta_n^{''})} \\ errorXY_{k,l}^{i,j} \leq 2^{-n(EXY_{k,l}^{i,j} - \delta_n^{''})} \end{gathered}$$

for some sequence $\delta_n^{''}(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|, M_1, M_2) \to 0$. Moreover, using definitions (2)-(6) and (34) we get:

$$\begin{gathered} EX_{k,l}^{i,j} \geq EX_{HL}^{i,j}, \quad EY_{k,l}^{i,j} \geq EY_{HL}^{i,j} \\ EXY_{k,l}^{i,j} \geq EXY_{HL}^{i,j}. \end{gathered}$$

Furthermore, using standard properties of types it follows that $th^{(i,j)} < 2^{-n(ETH(i,j) - \delta_n^{''})}$ where $ETH(i,j)$ is defined by

$$\min_{V_{UXYZ} \in \mathcal{O}^{i,j}} D(V_{Z|XYU}||W|V_{XYU}) + \mathrm{I}_V(X \wedge Y|U) \tag{35}$$

where

$$\mathcal{O}^{i,j} \triangleq \left\{\begin{array}{l} V_{UXYZ} : V_{UX} = P_U P_{X|U}^i,\, V_{UY} = P_U P_{Y|U}^j \\ \mathrm{I}_V(X \wedge Y, Z|U) - R_1^i \leq \eta_n \text{ or} \\ \mathrm{I}_V(Y \wedge X, Z|U) - R_2^j \leq \eta_n \text{ or} \\ \mathrm{I}_V(X \wedge Y \wedge Z|U) - R_1^i - R_2^j \leq \eta_n \end{array}\right\}$$

Using that $\min(EX_{HL}^{i,j}, EY_{k,l}^{i,j}, EY_{HL}^{i,j}) \leq ETH(i,j) + \eta_n$, the above inequalities prove part (i) of Theorem 1.

To prove part (ii) the following bound is useful:

$$\begin{aligned} Err_c(i,j) \leq & \frac{1}{N_1^i N_2^j} \sum_{a=1}^{N_1^i} \sum_{b=1}^{N_2^j} W^n \Big(\mathbf{z} : \exists k \in [M_1], \exists l \in [M_2], \\ & \exists c \in [N_1^k], \exists d \in [N_2^l] \,\text{such that}\, \mathbf{z} \notin O_{c,d}^{k,l} | \mathbf{x}_a^i, \mathbf{y}_b^j \Big). \end{aligned} \tag{36}$$

Using union bound, it is possible to expand (36) same way as $Err_c(i,j)$ is expanded in (27). Then the term corresponding to case $(k,c,l,d) = (i,a,j,b)$ can be upper bounded using standard properties of types. This upper bound leads to exponent $\min_{V_{UXYZ} \notin \mathcal{O}^{i,j}} D(V_{Z|XYU}||W|V_{XYU}) + I_V(X \wedge Y|U)$. The other cases can be upper bounded using the technique of [7] by exponent similar to (32)-(34), the sets on which the minimum is taken are different. Using the properties of these sets all terms within the positive part sign $|\ldots|^+$ can be lower bounded by η_n. Altogether, it can be seen that $Err_c(i,j)$ is small, if η_n goes to 0 appropriately slow.

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